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著者	Kikkawa Misako, Suzuki Tomonari
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COMMENTS ON SOME EXISTENCE THEOREMS OF BEST PROXIMITY POINTS FOR CONTRACTIVE-TYPE MAPPINGS

Misako Kikkawa and Tomonari Suzuki

Abstract

In 2010, Sadiq Basha proved two existence theorems of best proximity points for contractive-type mappings. The purpose of this paper is to clarify the mathematical structure of these theorems.

1. Introduction

Throughout this paper we denote by $\mathbf{N}$ the set of all positive integers and by $\mathbf{R}$ the set of all real numbers. We let (X, d) be a metric space and let A and B be nonempty subsets of X. Let T be a mapping from A into B and let S be a mapping from B into A. Define $d(A, B) \in \mathbf{R}$ and a function d^* from $X \times X$ into $[0, \infty)$ by

$$d(A, B) = \inf\{d(x, u) : x \in A,\ u \in B\}$$

and

$$d^*(a, b) = d(a, b) - d(A, B)$$

for any $a, b \in X$.

A point $x \in A$ is said to be a *best proximity point* of T if $d^*(x, Tx) = 0$ holds. Also, a point $u \in B$ is said to be a *best proximity point* of S if $d^*(Su, u) = 0$ holds. In the case where $A \cap B \neq \varnothing$, it is obvious that $d(A, B) = 0$ holds. Hence $x \in A$ is a fixed point of T iff x is a best proximity point of T. In the other case, where $A \cap B = \varnothing$, best proximity points of T are minimizers of the problem: $\min\{d(x, Tx) : x \in A\}$. Similarly for $y \in B$.

We human beings have studied the existence of best proximity points; see [3, 4, 5, 8, 10, 11] and others. In 2013, Sadiq Basha, Shahzad and Jeyaraj in [7] proved two existence theorems of best proximity points for Kannan-type and Chatterjea-type mappings. Very recently, in [9], the mathematical structure of these theorems were clarified.

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In 2010, Sadiq Basha [6] proved two existence theorems, Theorems 2 and 7 below, of best proximity points for contractive-type mappings. Motivated by the results in [9], in this paper, we clarify the mathematical structure of these theorems.

2. Banach contraction principle

The fixed point theorem for contractions is referred to as the *Banach contraction principle*. The proof of this is easy and well known. However, for the sake of completeness, we give a proof.

THEOREM 1 ([1, 2]). *Let (Y, d) be a metric space and let U be a contraction on Y, that is, there exists $r \in [0, 1)$ satisfying*

$$d(Ua, Ub) \leq rd(a, b) \tag{1}$$

for all $a, b \in Y$. Then the following hold:

(i) *$\{U^n a\}$ is a Cauchy sequence for all $a \in Y$.*
(ii) *U has at most one fixed point.*
(iii) *If Y is complete, then U has a unique fixed point.*
(iv) *If U has a fixed point c, then $\{U^n a\}$ converges to c for any $a \in Y$.*

PROOF. Fix $a \in Y$. We first show (i). We have

$$\sum_{j=1}^{\infty} d(U^j a, U^{j+1} a) \leq \sum_{j=1}^{\infty} r^j d(a, Ua) = \frac{r}{1-r} d(a, Ua) < \infty.$$

So, a standard argument shows that $\{U^n a\}$ is a Cauchy sequence.

In order to show (ii), we let $c, c' \in Y$ be fixed points of U. Then we have

$$d(c, c') = d(Uc, Uc') \leq rd(c, c').$$

Since $r < 1$, we have $d(c, c') = 0$. Thus, (ii) holds.

We next show (iii). By (i), we note that $\{U^n a\}$ is Cauchy. Since Y is complete, $\{U^n a\}$ converges to some $c \in Y$. We have

$$d(c, Uc) = \lim_{n\to\infty} d(U^n a, Uc) \leq \lim_{n\to\infty} rd(U^{n-1} a, c) = 0.$$

Hence $Uc = c$ holds, thus, c is a fixed point of U.

In order to prove (iv), we let $c \in Y$ be a fixed point of U. We have

$$\lim_{n\to\infty} d(U^n a, c) = \lim_{n\to\infty} d(U^n a, U^n c) \leq \lim_{n\to\infty} r^n d(a, c) = 0.$$

Thus, (iv) holds. □

3. Theorem 3.1 in [6]

In this section, we study Theorem 3.1 in [6], which is Theorem 2 in this paper. We begin with the notations and definitions that appear in the statement of Theorem 2.

Define two subsets A_0 and B_0 of A and B, respectively, by

$$A_0 = \{a \in A : d(a,b) = d(A,B) \text{ for some } b \in B\},$$

$$B_0 = \{b \in B : d(a,b) = d(A,B) \text{ for some } a \in A\}.$$

B is said to be *approximatively compact* with respect to A if every sequence $\{y_n\}$ in B satisfying the condition that $d(x, y_n) \to d(x, B)$ for some $x \in A$ has a convergent subsequence. T is said to be a *proximal contraction* if there exists $r \in [0, 1)$ such that

$$d(u, Tx) + d(Tx, Ty) + d(Ty, v) \le r d(x, y) \tag{2}$$

whenever x and y are distinct elements in A satisfying the condition that

$$d(u, Tx) = d(A, B) \qquad \text{and} \qquad d(v, Ty) = d(A, B) \tag{3}$$

for some $u, v \in A$.

Theorem 2 (Theorem 3.1 in [6]). *Assume the following:*
(a) *X is complete and A and B are closed.*
(b) *B is approximatively compact with respect to A.*
(c) *A_0 and B_0 are nonempty.*
(d) *$T(A_0) \subset B_0$.*
(e) *T is a proximal contraction.*

Then the following hold:
(i) *There exists a unique best proximity point z in A of T.*
(ii) *For each fixed $x_0 \in A_0$, there is a sequence $\{x_n\}_{n \in \mathbf{N} \cup \{0\}}$ in A such that $d(x_{n+1}, Tx_n) = d(A, B)$ for every $n \in \mathbf{N} \cup \{0\}$, where at least one of the x_n's is the same as z, or the sequence $\{x_n\}$ converges to z.*

It is important to confirm the following fact.

Lemma 3. *Assume* (c) *and* (d) *of Theorem* 2. *Then the following hold:*
(i) *For every $x \in A_0$, there exists $u \in A_0$ satisfying $d(u, Tx) = d(A, B)$.*
(ii) *For each fixed $x_0 \in A_0$, there is a sequence $\{x_n\}$ in A_0 such that $d(x_{n+1}, Tx_n) = d(A, B)$ for every $n \in \mathbf{N} \cup \{0\}$.*
(iii) *If $x \in A_0$ and $u \in A$ satisfy $d(u, Tx) = d(A, B)$, then $u \in A_0$ holds.*
(iv) *If a sequence $\{x_n\}_{n \in \mathbf{N} \cup \{0\}}$ in A satisfies $x_0 \in A_0$ and $d(x_{n+1}, Tx_n) = d(A, B)$ for $n \in \mathbf{N} \cup \{0\}$, then $x_n \in A_0$ holds for all $n \in \mathbf{N}$.*

Proof. (i), (iii) and (iv) obviously hold. (ii) follows from (i). □

We give a slight improvement of Theorem 2.

THEOREM 4. *Assume* (c)–(e) *of Theorem* 2. *Assume additionally* (a) *of Theorem* 2 *in the case where* $d(A,B)=0$. *Then the following hold:*

(i) *There exists a unique best proximity point* z *in* A *of* T.

(ii) *If a sequence* $\{x_n\}_{n\in\mathbf{N}\cup\{0\}}$ *in* A *satisfies* $x_0\in A_0$ *and* $d(x_{n+1},Tx_n)=d(A,B)$ *for* $n\in\mathbf{N}\cup\{0\}$, *then* $\{x_n\}$ *converges to* z.

Considering two cases of $d(A,B)>0$ and $d(A,B)=0$, we will prove Theorem 4.

LEMMA 5. *Assume* $d(A,B)>0$ *and* (c)–(e) *of Theorem* 2. *Then the following hold:*

(i) *If a sequence* $\{x_n\}_{n\in\mathbf{N}\cup\{0\}}$ *in* A *satisfies* $x_0\in A_0$ *and* $d(x_{n+1},Tx_n)=d(A,B)$ *for* $n\in\mathbf{N}\cup\{0\}$, *then there exists* $v\in\mathbf{N}$ *satisfying* $x_{v+1}=x_v$.

(ii) *There exists a unique element* $z\in A$ *satisfying* $d(z,Tz)=d(A,B)$.

(iii) *If* $d(x,Tz)=d(A,B)$ *for* $x\in A$, *then* $x=z$ *holds.*

(iv) *If a sequence* $\{x_n\}_{n\in\mathbf{N}\cup\{0\}}$ *in* A *satisfies* $x_0\in A_0$ *and* $d(x_{n+1},Tx_n)=d(A,B)$ *for* $n\in\mathbf{N}\cup\{0\}$, *then there exists* $v\in\mathbf{N}$ *satisfying* $x_n=z$ *for all* $n\geq v$.

PROOF. In order to prove (i), we let $\{x_n\}_{n\in\mathbf{N}\cup\{0\}}$ be a sequence in A satisfying $x_0\in A_0$ and $d(x_{n+1},Tx_n)=d(A,B)$ for $n\in\mathbf{N}\cup\{0\}$. By Lemma 3 (iv), we note $x_n\in A_0$ for all $n\in\mathbf{N}$. Arguing by contradiction, we assume $x_{n+1}\neq x_n$ for all $n\in\mathbf{N}$. Then since T is a proximal contraction, we have

$$\begin{aligned} d(x_n,x_{n+1}) &\leq d(x_n,Tx_{n-1})+d(Tx_{n-1},Tx_n)+d(Tx_n,x_{n+1}) \\ &\leq rd(x_{n-1},x_n)\leq\cdots\leq r^n d(x_0,x_1). \end{aligned}$$

Hence

$$\lim_{n\to\infty}(d(x_n,Tx_{n-1})+d(Tx_{n-1},Tx_n)+d(Tx_n,x_{n+1}))=0$$

holds. So we obtain

$$0<d(A,B)=\lim_{n\to\infty}d(Tx_n,x_{n+1})=0.$$

This is a contradiction. Therefore there exists $v\in\mathbf{N}$ satisfying $x_{v+1}=x_v$. We put $z=x_v$.

We next show (ii). Arguing by contradiction, we assume that there exists an element w of A satisfying

$$w\neq z \qquad \text{and} \qquad d(w,Tw)=d(A,B).$$

Since T is a proximal contraction, we have

$$d(w,z)\leq d(w,Tw)+d(Tw,Tz)+d(Tz,z)\leq rd(w,z).$$

Since $r \in [0,1)$, we obtain $d(w,z) = 0$ and hence $w = z$. This is a contradiction. Therefore we have shown (ii).

In order to show (iii), suppose $d(x,Tz) = d(A,B)$ for some $x \in A$. Arguing by contradiction, we assume $x \neq z$. Then we have $x \in A_0$ and hence $Tx \in B_0$. So there exists $u \in A_0$ satisfying $d(u,Tx) = d(A,B)$. Since T is a proximal contraction, we have

$$\begin{aligned} 2d(A,B) &\leq 2d(A,B) + d(Tz,Tx) \\ &= d(x,Tz) + d(Tz,Tx) + d(Tx,u) \\ &\leq rd(z,x) \\ &\leq r(d(z,Tz) + d(Tz,x)) \\ &= 2rd(A,B). \end{aligned}$$

Hence, $d(A,B) = 0$ holds. This is a contradiction. Therefore we obtain (iii).

In order to prove (iv), we let $\{x_n\}_{n \in \mathbf{N} \cup \{0\}}$ be a sequence in A satisfying $x_0 \in A_0$ and $d(x_{n+1}, Tx_n) = d(A,B)$ for $n \in \mathbf{N} \cup \{0\}$. From (i), there exists $v \in \mathbf{N}$ satisfying $x_v = z$. By (iii), we have $x_{v+1} = z$. Thus, we obtain $x_n = z$ for all $n \in \mathbf{N}$ with $n \geq v$. □

Lemma 6. *Assume $d(A,B) = 0$,* (a) *and* (c)–(e) *of Theorem* 2. *Then the following hold:*

- (i) *$A_0 = B_0 = A \cap B$ holds.*
- (ii) *A_0 is complete.*
- (iii) *The restriction U of T to A_0 is a contraction on A_0.*
- (iv) *There exists a unique element $z \in A_0$ satisfying $Uz = z$.*
- (v) *z is a unique element of A satisfying $d(z,Tz) = d(A,B)$.*
- (vi) *If a sequence $\{x_n\}_{n \in \mathbf{N} \cup \{0\}}$ in A satisfies $x_0 \in A_0$ and $d(x_{n+1}, Tx_n) = d(A,B)$ for $n \in \mathbf{N} \cup \{0\}$, then $x_n = U^n x_0$ holds for all $n \in \mathbf{N}$ and $\{x_n\}$ converges to z.*

Proof. (i) obviously holds.

We next show (ii). Since A and B are closed, A_0 is closed. Since X is complete, A_0 is complete.

In order to prove (iii), we let U be the restriction of T to A_0. Fix $x, y \in A_0$. It is obvious that $Ux = Tx \in B_0 = A_0$ holds. So U is a mapping on A_0. Put $u = Tx$ and $v = Ty$. Then

$$d(u,Tx) = d(v,Ty) = 0 = d(A,B)$$

holds. In the case where $x \neq y$, since T is a proximal contraction, we have

$$d(Ux,Uy) = d(u,Tx) + d(Tx,Ty) + d(Ty,v) \leq rd(x,y).$$

It the other case, where $x = y$, it is obvious that $d(Ux,Uy) = 0 \leq rd(x,y)$ holds. Therefore we have shown that U is a contraction on A_0.

(iv) follows from Theorem 1.

We next show (v). We have

$$d(z, Tz) = d(z, Uz) = 0 = d(A, B).$$

Arguing by contradiction, we assume that there exists an element w of A satisfying

$$w \neq z \qquad \text{and} \qquad d(w, Tw) = d(A, B).$$

Then we have $w \in A_0$. Hence w is a fixed point of U. This is a contradiction. Therefore we have shown (v).

In order to prove (vi), we let $\{x_n\}_{n \in \mathbf{N} \cup \{0\}}$ be a sequence in A satisfying $x_0 \in A_0$ and $d(x_{n+1}, Tx_n) = d(A, B)$ for $n \in \mathbf{N} \cup \{0\}$. By Lemma 3 (iv), we note that $\{x_n\}$ is a sequence in A_0. We have

$$d(x_{n+1}, Ux_n) = d(x_{n+1}, Tx_n) = d(A, B) = 0$$

for $n \in \mathbf{N}$. Thus, we obtain $x_n = U^n x_0$. By Theorem 1, $\{x_n\}$ converges to z. □

4. Theorem 3.3 in [6]

In this section, we study Theorem 3.3 in [6], which is Theorem 7 in this paper.

THEOREM 7 (Theorem 3.3 in [6]). *Assume the following:*

(a) *X is complete and A and B are closed.*

(b) *S is nonexpansive, that is, $d(Su, Sv) \leq d(u, v)$ for any $u, v \in B$.*

(c) *T is a contraction with contraction constant r.*

(d) *If $(x, y) \in A \times B$ satisfies $d(A, B) < d(x, y)$, then $d(Sy, Tx) < d(x, y)$ holds.*

Define a sequence $\{a_n\}_{n \in \mathbf{N} \cup \{0\}}$ by $a_0 \in A$, $a_{2n+1} = Ta_{2n}$ and $a_{2n+2} = Sa_{2n+1}$ for $n \in \mathbf{N} \cup \{0\}$. Then the following hold:

(i) *There exist $z \in A$ and $w \in B$ satisfying $d(z, Tz) = d(A, B)$, $d(Sw, w) = d(A, B)$ and $d(z, w) = d(A, B)$.*

(ii) *$\{a_{2n}\}$ and $\{a_{2n+1}\}$ converge to some best proximity points in A and B of T and S, respectively.*

(iii) *If $x, y \in A$ are best proximity points in A of T, then*

$$d(x, y) \leq \frac{2}{1 - r} d(A, B)$$

holds.

We give a slight improvement of Theorem 7.

THEOREM 8. *Assume the following:*

(a) *Either A or B is complete.*

(b) *S is nonexpansive.*

(c) *T is a contraction with contraction constant r.*

(d) *$d^*(x, Tx) > 0$ implies $d^*(STx, Tx) \neq d^*(x, Tx)$.*

Define a sequence $\{a_n\}_{n \in \mathbf{N} \cup \{0\}}$ by $a_0 \in A$, $a_{2n+1} = Ta_{2n}$ and $a_{2n+2} = Sa_{2n+1}$ for $n \in \mathbf{N} \cup \{0\}$. Then the following hold:

(i) *ST and TS are contractions on A and B, respectively.*

(ii) *ST and TS have unique fixed points $z \in A$ and $w \in B$, respectively.*

(iii) *z and w are best proximity points in A and B of T and S, respectively, which satisfy $Tz = w$ and $Sw = z$.*

(iv) *$\{a_{2n}\}$ and $\{a_{2n+1}\}$ converge to z and w, respectively.*

(v) *If $x, y \in A$ are best proximity points of T, then*

$$d(x, y) \leq \frac{2}{1-r} d(A, B)$$

holds.

(vi) *If $x \in A$ is a best proximity point of T, then*

$$d(z, x) \leq \frac{2}{1-r} d(A, B) \qquad \text{and} \qquad d(x, w) \leq \frac{1+r}{1-r} d(A, B)$$

hold.

Remark. It is obvious that (a) of Theorem 8 is weaker than (a) of Theorem 7. It is also obvious that (d) of Theorem 8 is weaker than (d) of Theorem 7.

Proof. We first show (i). For $x, y \in A$ and $u, v \in B$, we have

$$d(STx, STy) \leq d(Tx, Ty) \leq rd(x, y)$$

and

$$d(TSu, TSv) \leq rd(Su, Sv) \leq rd(u, v),$$

thus, ST and TS are contractions with contraction constant r.

We next prove (ii). We consider the following two cases:

- A is complete.
- B is complete.

In the first case, by Theorem 1 (iii), ST has a unique fixed point $z \in A$. Since

$$TS(Tz) = T(STz) = Tz,$$

$w := Tz$ is a fixed point of TS. By Theorem 1 (ii), w is a unique fixed point of TS. In the second case, by Theorem 1 (iii), TS has a unique fixed point $w \in B$. Since $STSw = Sw$, $z := Sw$ is a fixed point of ST. By Theorem 1 (ii), z is a unique fixed point of ST.

Let us prove (iii). We have already shown $Tz = w$ and $Sw = z$. It follows from (d) and $STz = z$ that $d^*(STz, Tz) = d^*(z, Tz) = 0$ holds. Thus, z is a best proximity point in A of T. Since

$$0 = d^*(STz, Tz) = d^*(Sw, w),$$

w is a best proximity point in B of S. We have proved (iii).

It is obvious that (iv) follows from Theorem 1 (iv).

Let us prove (v). Let $x, y \in A$ be best proximity points of T. Then we have

$$\begin{aligned} d(x, y) &\le d(x, Tx) + d(Tx, Ty) + d(Ty, y) \\ &\le d(x, Tx) + rd(x, y) + d(y, Ty) \\ &= rd(x, y) + 2d(A, B). \end{aligned}$$

Hence (v) holds.

We finally prove (vi). Let $x \in A$ be a best proximity point of T. Since z is also a best proximity point of T, we have from (v)

$$d(z, x) \le \frac{2}{1-r} d(A, B).$$

We also have

$$\begin{aligned} d(x, w) &\le d(x, Tx) + d(Tx, w) \\ &= d(x, Tx) + d(Tx, Tz) \\ &\le d(A, B) + rd(x, z) \\ &\le \left(1 + \frac{2r}{1-r}\right) d(A, B) \\ &= \frac{1+r}{1-r} d(A, B). \end{aligned}$$

Thus, (vi) holds. □

The following examples tell that three numbers that appear in (v) and (vi) of Theorem 8 are best possible.

Example 9. Let $r \in (0, 1)$ and put $\sigma := 2/(1-r) \in (2, \infty)$. Define sequences $\{x_n\}_{n \in \mathbf{N} \cup \{0\}}$ and $\{u_n\}_{n \in \mathbf{N}}$ by

$$x_n = (0, \sigma r^n) \qquad \text{and} \qquad u_n = (1, \sigma r^n).$$

Put $z = (0, 0)$ and $w = (1, 0)$. Define subsets A, B and X of $\mathbf{R}^2$ by

$$A = \{z\} \cup \{x_n : n \in \mathbf{N} \cup \{0\}\},$$
$$B = \{w\} \cup \{u_n : n \in \mathbf{N}\}$$

and $X = A \cup B$. Define mappings T and S by

$$Tx_n = u_{n+1}, \qquad Tz = w,$$
$$Su_n = x_n, \qquad Sw = z.$$

Define a function e from $X \times X$ into $[0, \infty)$ by

$$e(a,b) = \begin{cases} 1 & \text{if } (a,b) = (x_0, u_1) \text{ or } (a,b) = (u_1, x_0) \\ \|a - b\|_1 & \text{otherwise,} \end{cases}$$

where $\|\cdot\|_1$ is the ℓ_1-norm on $\mathbf{R}^2$. Define a function d from $X \times X$ into $[0, \infty)$ by

(4) $$d(a,b) = \min\left\{\sum_{j=1}^{n} e(a_{j-1}, a_j) : (a_0, \ldots, a_n) \in X^{n+1},\ a_0 = a,\ a_n = b\right\}.$$

Then the following hold:

(i) A, B and X are complete.
(ii) S is nonexpansive.
(iii) T is a contraction.
(iv) (d) of Theorem 8 holds.
(v) x_0 and z are best proximity points of T.
(vi) $d(A,B) = 1$.
(vii) $d(z, x_0) = \dfrac{2}{1-r}$.
(viii) $d(x_0, w) = \dfrac{1+r}{1-r}$.

Proof. We first note

$$d(x_0, x) = e(x_0, x) = \|x_0 - x\|_1,$$
$$d(x_0, u) = e(x_0, u_1) + e(u_1, u) = \|x_0 - u\|_1 - 2,$$
$$d(x, y) = e(x, y) = \|x - y\|_1,$$
$$d(u, v) = e(u, v) = \|u - v\|_1$$

for $x, y \in A \setminus \{x_0\}$ and $u, v \in B$; see also Lemma 12 below. So, (vii) and (viii) hold.

(i) obviously holds.

Since

$$d(Su, Sv) = d(u, v)$$

for any $u, v \in B$, (ii) holds.

Since

$$d(Tx, Ty) = rd(x, y)$$

for any $x, y \in A$, (iii) holds.

Since $d^*(STx, Tx) = 0$ for any $x \in A$, (iv) holds.

(v) and (vi) obviously hold. □

We show that even in the case where $r = 0$, three numbers that appear in (v) and (vi) of Theorem 8 are best possible.

EXAMPLE 10. Put $r = 0$, $\sigma = 2$ and

$$x_0 = (0, 2), \qquad z = (0, 0), \qquad w = (1, 0).$$

Define subsets A, B and X of $\mathbf{R}^2$ by

$$A = \{x_0, z\}, \qquad B = \{w\}, \qquad X = A \cup B.$$

Define mappings T and S by

$$Tx_0 = w, \qquad Tz = w, \qquad Sw = z.$$

Define a function e from $X \times X$ into $[0, \infty)$ by

$$e(a, b) = \begin{cases} 1 & \text{if } (a, b) = (x_0, w) \text{ or } (a, b) = (w, x_0) \\ \|a - b\|_1 & \text{otherwise.} \end{cases}$$

Define a function d from $X \times X$ into $[0, \infty)$ by (4). Then (i)–(viii) of Example 9 hold.

5. Lemma

In this section, we prove one lemma, connected with the underlying metric spaces in Examples 9 and 10. See also Examples 10 and 13 in [9].

We give the definition of metric space, though it is well known. Let X be a nonempty set and let d be a function from $X \times X$ into $[0, \infty)$. Then (X, d) is said to be a *metric space* if the following hold:

(D1) $d(x, x) = 0$

(D2) $d(x, y) = 0 \Rightarrow x = y$

(D3) $d(x, y) = d(y, x)$

(D4) $d(x, z) \le d(x, y) + d(y, z)$

We can easily prove the following.

Lemma 11. *Let X be a nonempty set and let e be a function from $X \times X$ into $[0, \infty)$ satisfying* (D1) *and* (D3) *with $d := e$. Define a function d from $X \times X$ into $[0, \infty)$ by*

$$d(x, y) = \inf\left\{\sum_{j=1}^{n} e(u_{j-1}, u_j) : (u_0, \dots, u_n) \in X^{n+1},\ u_0 = x,\ u_n = y\right\}.$$

Assume (D2). *Then (X, d) is a metric space.*

We finally prove the following.

Lemma 12. *Let (X, ρ) be a metric space and let A and B be nonempty subsets of X. Put $Y := A \cup B$ and $\ell := \rho(A, B) \in (0, \infty)$. Assume that there exist a subset A_2 of A and mappings Q and R from A_2 into A and B, respectively, satisfying*

$$\rho(a, v) = 2\ell + \rho(a, Qv), \tag{5}$$

$$\rho(a, Rv) = \ell + \rho(a, Qv), \tag{6}$$

$$\rho(v, b) = 3\ell + \rho(Rv, b), \tag{7}$$

$$\rho(Qv, b) = \ell + \rho(Rv, b) \tag{8}$$

for any $a \in A$, $b \in B$, $v \in A_2$ with $a \neq v$. Put $A_1 = A \backslash A_2$. Define a function e from $Y \times Y$ into $[0, \infty)$ by

$$e(v, Rv) = e(Rv, v) = \ell \qquad \text{for all } v \in A_2,$$

$$e(x, y) = \rho(x, y) \qquad \text{otherwise}.$$

Define a function d from $Y \times Y$ into $[0, \infty)$ by

$$d(x, y) = \min\left\{\sum_{j=1}^{n} e(u_{j-1}, u_j) : (u_0, \dots, u_n) \in Y^{n+1},\ u_0 = x,\ u_n = y\right\}.$$

Then the following hold:

(i) $Qv \in A_1$ *for all* $v \in A_2$.
(ii) $e(x, y) \le e(x, v) + e(v, y)$ *for* $x, y \in Y$ *and* $v \in A_2$.
(iii) $e(x, y) \le e(x, z) + e(z, y)$ *for* $x, y, z \in Y$ *with* $(x, z), (y, z) \notin \mathrm{Gr}(R)$, *where* $\mathrm{Gr}(R)$ *is the graph of* R.
(iv) $e(x, y) \le e(x, z) + e(z, y)$ *for* $x, y \in A_1 \cup B$ *and* $z \in Y$.
(v) $d(x, y) = \rho(x, y)$ *for* $x, y \in A_1 \cup B$.
(vi) $d(u, v) = \rho(u, v)$ *for* $u \in A_1$ *and* $v \in A_2$.
(vii) $d(v, b) = \rho(v, b) - 2\ell = \ell + \rho(Rv, b)$ *for* $v \in A_2$ *and* $b \in B$.
(viii) $d(v, v') = \rho(v, v') - 2\ell = 2\ell + \rho(Rv, Rv')$ *for* $v, v' \in A_2$ *with* $v \neq v'$.
(ix) $d(A, B) = \ell$.
(x) (X, d) *is a metric space.*

Remark. $A_2 = \varnothing$ is possible. On the other hand, $A_2 = A$ cannot be possible from (i).

Proof. We first redefine d by

$$d(x,y) = \inf\left\{\sum_{j=1}^{n} e(u_{j-1},u_j) : (u_0,\dots,u_n) \in Y^{n+1},\ u_0 = x,\ u_n = y\right\}.$$

After showing (viii), we will find that the above infimum is the minimum.

We have by (7)

$$e(v,Rv) = \ell < 3\ell = \rho(v,Rv) \tag{9}$$

for all $v \in A_2$. So we note

$$e(x,y) \le \rho(x,y)$$

for all $x, y \in Y$. It is obvious that

$$e(x,x) = \rho(x,x) = 0 \qquad \text{and} \qquad e(x,y) = e(y,x)$$

hold for all $x \in Y$. Thus, (D1) and (D3) with $d := e$ hold.

We will show (i). Arguing by contradiction, we assume that $Qv \in A_2$ for some $v \in A_2$. Then we have by (7) and (8)

$$3\ell \le 3\ell + \rho(RQv,Rv) = \rho(Qv,Rv) = \ell < 3\ell,$$

which implies a contradiction. Therefore we obtain (i).

In order to show (ii), we let $v \in A_2$. We observe the following:

$$\begin{aligned}
e(v,v) + e(v,Rv) &= e(v,Rv)\\
e(a,v) + e(v,Rv) &= \rho(a,v) + \ell = 3\ell + \rho(a,Qv)\\
&= 2\ell + \rho(a,Rv) \ge \rho(a,Rv) \ge e(a,Rv),\\
e(b,v) + e(v,Rv) &= \rho(b,v) + \ell = 4\ell + \rho(b,Rv)\\
&\ge \rho(b,Rv) = e(b,Rv),\\
e(Rv,v) + e(v,Rv) &= 2\ell \ge 0 = e(Rv,Rv)
\end{aligned}$$

for $a \in A$, $b \in B$, $v \in A_2$ with $a \ne v$ and $b \ne Rv$. So (ii) holds in the case where $Rv \in \{x,y\}$. In the other case, where $Rv \notin \{x,y\}$, we have

$$e(x,y) \le \rho(x,y) \le \rho(x,v) + \rho(v,y) = e(x,v) + e(v,y).$$

We have shown (ii). So we note

$$d(x, y) = \inf\left\{\sum_{j=1}^{n} e(u_{j-1}, u_j) : (u_1, \dots, u_{n-1}) \in (A_1 \cup B)^{n-1},\ u_0 = x,\ u_n = y\right\}.$$

In order to show (iii), we let $x, y, z \in Y$ satisfy $(x, z), (y, z) \notin \mathrm{Gr}(R)$. We have already shown (iii) in the case where $z \in A_2$. So suppose $z \in A_1 \cup B$. Then we have $(z, x), (z, y) \notin \mathrm{Gr}(R)$ and hence

$$e(x, y) \le \rho(x, y) \le \rho(x, z) + \rho(z, y) = e(x, y) + e(z, y).$$

We have shown (iii). In particular, (iv) holds. So we note

$$\begin{aligned} d(x, y) = \min\{&e(x, y), \inf\{e(x, z) + e(z, y) : z \in A_1 \cup B\}, \\ &\inf\{e(x, z) + e(z, w) + e(w, y) : z, w \in A_1 \cup B\}\}. \end{aligned}$$

Using (ii)–(iv), we will prove (v)–(viii). We can easily prove (v). For $u \in A_1$ and $v \in A_2$, we have

$$e(v, Rv) + e(Rv, u) = \ell + \rho(Rv, u) = 2\ell + \rho(Qv, u) = \rho(v, u) = e(v, u),$$

which implies (vi). For $b \in B$ and $v \in A_2$ with $b \neq Rv$, we have

$$e(v, Rv) + e(Rv, b) = \ell + \rho(Rv, b) = \rho(v, b) - 2\ell = e(v, b) - 2\ell.$$

We also have by (9)

$$e(v, Rv) = \ell = \rho(v, Rv) - 2\ell.$$

These imply (vii). For $v, v' \in A_2$ with $v \neq v'$, we further observe the following.

$$\begin{aligned} e(v, Rv) + e(Rv, Rv') + e(Rv', v') &= 2\ell + \rho(Rv, Rv') \\ &= \rho(Qv, Rv') + \ell = \rho(Qv, Qv') + 2\ell = \rho(v, Qv') \\ &= \rho(v, v') - 2\ell =: \eta, \\ e(v, Rv) + e(Rv, v') = \ell + \rho(Rv, v') &= 4\ell + \rho(Rv, Rv') \ge \eta \qquad (Rv \neq Rv'), \\ e(v, Rv') + e(Rv', v') = \rho(v, Rv') + \ell &= 4\ell + \rho(Rv, Rv') \ge \eta \qquad (Rv \neq Rv'), \\ e(v, v') = \rho(v, v') &\ge \eta. \end{aligned}$$

From these observations, we obtain (viii).

Let us prove (ix). Since $d(x, y) \le e(x, y) \le \rho(x, y)$ holds for $x, y \in Y$, we have $d(A, B) \le \rho(A, B) = \ell$. Fix $(a, b) \in A \times B$. In the case $a \in A_1$, we have by (v)

$$\ell = \rho(A, B) \le \rho(a, b) = d(a, b).$$

In the other case, where $a \in A_2$, we have by (vii)

$$\ell \le \ell + \rho(Ra, b) = d(a, b).$$

Thus, we obtain (ix).

From (v)–(viii), we obtain (D2). So by Lemma 11, (X, d) is a metric space. $\square$

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(M. Kikkawa)
Department of Mathematics
Faculty of Science, Saitama University
Sakura, Saitama 338-8570, *Japan*
E-mail: mi-sa-ko-kikkawa@jupiter.sannet.ne.jp

(T. Suzuki)
Department of Basic Sciences
Faculty of Engineering
Kyushu Institute of Technology
Tobata, Kitakyushu 804-8550, *Japan*
E-mail: suzuki-t@mns.kyutech.ac.jp